JOURNAL OF ALGEBRA 114, 484–488 (1988)

Cohen–Macaulay Simplicial Complexes*

MARY L. THOMPSON

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104-3859

Communicated by David Buchsbaum

Received May 15, 1986

Let K be a field, and let Δ be a finite simplicial complex with vertices $x_0, x_1, ..., x_n$, which we will regard as indeterminates over the field K. Let I_Δ be the ideal in $R = K[x_0, ..., x_n]$ generated by all monomials $x_{i_0}x_{i_1}\cdots x_{i_r}$, $i_0 < i_1 < \cdots i_k$, for which $\{x_{i_0}, ..., x_{i_r}\}$ is not a simplex in Δ. Let $L[\Delta] = R/I_\Delta$. The simplicial complex Δ is said to be Cohen–Macaulay (which we will abbreviate C-M) over K precisely when the ring $K[\Delta]$ is Cohen–Macaulay.

G. Reisner has characterized the simplicial complexes which are Cohen–Macaulay over K:

THEOREM [R]. *Δ is Cohen–Macaulay over K if and only if $\tilde{H}^i(\text{link } \sigma, K) = 0$, $i < \dim(\text{link } \sigma)$ for every face σ of Δ. Here,* link $\sigma = \{\tau \in \Delta : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \varnothing\}$. *Since* link $(\varnothing) = \Delta$, *for Δ to be C-M, we must have* $\tilde{H}^i(\Delta, K) = 0$, $i < \dim \Delta$.

Our main result is a proof of the "if" part of Reisner's theorem in much greater generality and by a completely different method. We use the condition on cohomology of the links to show that certain rings associated to Δ are Cohen–Macaulay. All rings considered in this paper are commutative, noetherian, and have 1.

1.1. THEOREM. *Let K be a field, R a finitely generated K-algebra, Δ a simplicial complex Cohen–Macaulay over K. Let I be a family of ideals of R, indexed by the subcomplexes of Δ, satisfying*

(i) $I_{\Delta_1 \cup \Delta_2} = I_{\Delta_1} \cap I_{\Delta_2}$,

(ii) $I_{\Delta_1 \cap \Delta_2} = I_{\Delta_1} + I_{\Delta_2}$.

* The results in this paper are part of the author's doctoral thesis, University of Michigan, written under the direction of Melvin Hochster.

0021-8693/88 $3.00

(iii) *There is a constant* c *so that if* σ *is a simplex in* Δ, R/I_σ *is C-M of dimension equal to* $\dim(\sigma)+c$.

Then R/I_Δ *is C-M.*

To see that this theorem reproves the "if" part of Reisner's theorem, let $R=K[x_0, ..., x_n]$, where the x_i are vertices of Δ. If Δ' is a subcomplex, let $I_{\Delta'}$ be the ideal generated by all monomials $x_{i_0}\cdots x_{i_r}$, where $\{x_{i_0}, ..., x_{i_r}\}$ is not a simplex of Δ'. In this setup, if σ is a simplex, R/I_σ is a polynomial ring of dimension equal to $\dim(\sigma)+1$.

Our methods yield a version of Theorem 1.1 in which there is no field in sight.

1.2. THEOREM. *Let* Δ *be a simplicial complex satisfying: for every face* σ *of* Δ, $\tilde{H}^i$ (link σ) $=0$, $i<\dim(\text{link }\sigma)$, *and* $\tilde{H}^{\dim(\text{link }\sigma)}(\text{link }\sigma)$ *is torsion free over Z. If R is a ring and I a family of ideals of R satisfying conditions* (i), (ii), *and* (iii) *of Theorem* 1.1, *then* R/I_Δ *is Cohen–Macaulay.*

In either theorem, it is not hard to show that R/I_Δ is C-M for Δ belonging to a certain subclass of the C-M simplicial complexes, the constructible simplicial complexes. These are defined recursively as follows: All simplices are constructible, as are all 1-dimensional connected simplicial complexes, and if $\Delta=\Delta_1\cup\Delta_2$, where Δ_1, Δ_2 and $\Delta_1\cap\Delta_2$ are constructible of dimensions n, n and $n-1$, then Δ is constructible. All 0-dimensional simplicial complexes are constructible, as are all 1-dimensional connected simplicial complexes; these are precisely the Cohen–Macaulay simplicial complexes of dimensions 0 and 1. There are, however, C-M complexes that are not constructible [Sta].

Let Δ be a finite simplicial complex of dimension d with maximal faces of $\sigma_1, \sigma_2, ..., \sigma_N$. There is an open cover of $|\Delta|$, $\{U_1, ..., U_N\}$, with the properties: all intersections $U_{i_0}\cap\cdots\cap U_{i_r}$ are empty or contractible, $\sigma_i\subset U_i$ for each i, and $U_{i_0}\cap\cdots\cap U_{i_r}=\varnothing$ if and only if $\sigma_{i_0}\cap\cdots\cap\sigma_{i_r}=\varnothing$. For example, representing each $x\in|\Delta|$ as $x=\sum a_ix_i$, where each $a_i>0$, $\sum a_i=1$, and $\{x_i: a_i>0\}$ is a face of Δ containing x in its interior, one can take $U_i=\{\sum a_jx_j: \sum_{x_j\in\sigma_i}a_j>1-1/(n+1)\}$ (where Δ has $n+1$ vertices). We may compute the reduced Čech cohomology of Δ with coefficients in S, a ring, using this open cover; this will coincide with the reduced simplicial cohomologyof Δ with coefficients in S. The next lemma is a consequence of this discussion.

2.1. LEMMA. *The reduced simplicial cohomology of* Δ *with coefficients in S may be computed by taking the cohomology of the complex*

$$O\to S\to\bigoplus_{i=1}^{N} Se_i\to\bigoplus_{\substack{1\leqslant i<j\leqslant N\\ \sigma_i\cap\sigma_j\neq\varnothing}} Se_{ij}\to\cdots.$$

Now suppose R is a ring and I is a family of ideals satisfying conditions (i) and (ii) of Theorem 1.1, so that I is a distributive family of ideals. Let $I_1, ..., I_N$ be the ideals in I corresponding to the maximal faces of Δ, $\sigma_1, \sigma_2, ..., \sigma_N$. We will define certain complexes of R-modules.

Let C_{-1} be the complex

$$0 \to \bigcap_{j=1}^{N} I_j \to \bigoplus_{j=1}^{N} R/I_j \to \bigoplus_{1 \leqslant j \leqslant k \leqslant N} \frac{R}{I_j + I_k} \to \cdots.$$

Here, $C_{-1}^{-1} = R/I_j$ and $C_{-1}^m m = \bigoplus_{1 \leqslant j_0 < \cdots < j_m \leqslant N} R/(I_{j_0} + \cdots + I_{j_m})$. This complex is obtained by tensoring the complexes $0 \to R \to R/I_j$ together, when R is of degree 0 and R/I_j of degree 1, then shifting degrees down one and replacing R in the -1 spot by $R/I_j = R/I_\Delta$. C_{-1} is exact; see [EH].

Let D_n, $0 \leqslant n \leqslant \dim \Delta$ be the subcomplex of C_{-1} with

$$D_r^m = \bigoplus_{\dim \sigma_{j_0} \cap \cdots \cap \sigma_{j_m} \leqslant n} \frac{R}{I_{j0} + \cdots + I_{j_m}}.$$

We now define complexes C_n and K_n, $-1 \leqslant n \leqslant \dim \Delta$, by $C_n = C_{-1}/D_{n-1}$, $K_n = D_n/D_{n-1} \subset C_n$. Note that $C_n/K_n \cong C_{n+1}$. As modules, $C_n^{-1} = R/I_j$ and

$$C_n^m = \bigoplus_{\substack{1 \leqslant j_0 < \cdots < j_m \leqslant N \\ \dim \sigma_{j_0} \cap \cdots \cap \sigma_{j_m} \geqslant n}} \frac{R}{I_{j0} + \cdots + I_{j_m}},$$

$K_n^{-1} = 0$ and

$$K_n^m = \bigoplus_{\substack{1 \leqslant j_0 < \cdots < j_m \leqslant N \\ \dim \sigma_{j_0} \cap \cdots \cap \sigma_{j_m} = n}} \frac{R}{I_{j0} + \cdots + I_{j_m}}.$$

We can represent K_n as a direct sum of complexes: $K_n = \bigoplus_{\dim \sigma = n} K_\sigma$, where

$$K_\sigma^m = \bigoplus_{\substack{1 \leqslant j_0 < \cdots < j_m \leqslant N \\ \sigma_{j_0} \cap \cdots \cap \sigma_{j_m} = \sigma}} \frac{R}{I_{j0} + \cdots + I_{j_m}},$$

a free module over R/I_σ, since $I_{j_0} + \cdots + I_{j_m} \times I_\sigma$.

2.2. Lemma. *$H^i(K_\sigma) \cong \tilde{H}^{i-1}(L, R/I_\sigma)$ for all i, where $L = \operatorname{link} \sigma$, σ a face of Δ.*

Proof. The maximal faces of L are in one-to-one corespondence with the maximal faces of Δ containing $\sigma: \sigma \subset \sigma_i$ if and only if $\sigma_i - \sigma$ is a maximal face of L. Let $S = R/I_\sigma$, and let C_σ be the complex

$$0 \to S \to \bigoplus_{\sigma \subset \sigma_i} Se_i \to \bigoplus_{\substack{i<j \\ \sigma \subset \sigma_i \cap \sigma_j}} Se_{ij} \to \cdots$$

$$= \bigoplus_{\sigma \subset \sigma_i} (0 \to S \to Se_i \to 0).$$

This complex is exact, with K_σ as a subcomplex. The quotient C_σ/K_σ is

$$0 \to S \to \bigoplus_{\sigma \subsetneq \sigma_i} Se_i \to \bigoplus_{\substack{i<j \\ \sigma \subsetneq \sigma_i \cap \sigma_j}} Se_{ij} \to \cdots .$$

According to Lemma 2.1, taking cohomology of C_σ/K_σ gives the reduced simplicial cohomology of Δ with coefficients in S. Considering the long exact sequence corresponding to the short exact sequence $0 \to K_\sigma \to C_\sigma \to C_\sigma/K_\sigma \to 0$ and using $H^i(C_\sigma) = 0$ gives $H^i(K_\sigma) \cong \tilde{H}^i(L, R/I_\sigma)$, all i.

2.3. Corollary. *If R is a K-algebra, K is a field, and Δ is C-M over K, then $H^i(K_\sigma) = 0$ if $i \neq d - \dim \sigma$, and $H^{d-\dim\sigma}(K_\sigma)$ is a free R/I_σ module.*

2.4. Corollary. *If $\tilde{H}^i(L) = 0$, $i < \dim L$, and $\tilde{H}^{\dim L}(L)$ is a torsion free Z-module, then $H^i(K_\sigma) = 0$, $i \neq d - \dim \sigma$, and $H^{d-\dim\sigma}(K_\sigma)$ is a free R/I_σ module. (Here, $L = \mathrm{link}(\sigma)$.)*

These corollaries follow from Lemma 2.2 and an appropriate universal coefficient theorem.

2.5. Lemma. *Suppose $H^i(K_\sigma) = 0$ if $i \neq d - \dim \sigma$ for every face σ of Δ. Then the map $H^{d-n}(K_n) \to H^{d-n}(C_n)$ induced by the inclusion $K_n \to C_n$ is surjective, $-1 \leqslant n \leqslant d$.*

Proof. Let $m = d - n$. Let x be a cycle in C_n^m, and write $\bar{x} = \bar{x}_{n+r} + \cdots + \bar{x}_n$, where $\bar{}$ denotes the image of x in $H(C_n)$, and where $x_i \in \bigoplus_{\dim \sigma_{i_0} \cap \cdots \cap \sigma_{i_m} = i} R/(I_{i_0} + \cdots + I_{i_m})$. Suppose that r is the least non-neagative integer for which this is possible. If $r = 0$, then $\bar{x} \in \mathrm{im}(H^m(K_n) \to H^m(C_n))$. Suppose $r > 0$. Then $x_{n+r} = \sum x_r$, $x_r \in \bigoplus_{\sigma_{i_0} \cap \cdots \cap \sigma_{i_m} = \sigma} R/(I_{i_0} + \cdots + I_{i_m})$. Regarded as an element of K_σ, $dx_\sigma = 0$. But $H^m(K_\sigma) = 0$ since $m \neq n - (d + r)$, so $x_\sigma = dy_\sigma$, $y_\sigma \in K_\sigma^{m-1}$. Regard y_σ as an element of C_n^{m-1}. Then $dy_\sigma = x_\sigma - x'_\sigma$, where $x'_\sigma \in \bigoplus_{\dim \sigma_{i_0} \cap \cdots \cap \sigma_{i_m} < n+r} R/(I_{i_0} + \cdots + I_{i_m})$. Going through this procedure for all σ which occur, we may substitute and obtain another expression for $\bar{x}$, showing that r was not minimal.

2.6. LEMMA. *Suppose Δ, R, K are as in Theorem* 1.1, *or suppose Δ, R are as in Theorem* 1.2. *Then $H^i(C_n)=0$ if $i \neq d-n$, and $H^{d-n}(C_n)$ is* 0 *or a Cohen–Macaulay R-module of dimension $n+c-1$, $0 \leqslant n \leqslant d$.*

Proof. We use induction. If $n=0$, there is an exact sequence $0 \to K_{-1} \to C_{-1} \to C_0 \to 0$. Since C_{-1} is exact, we get $H^1(C_0) \cong H^{i+1}(K_{-1})$ for all i. From the corollaries, $H^{i+1}(K_{-1})=0$ if $i+1 \neq d+1$ and $H^{d+1}(K_{-1})$ is a free $R/I_\varnothing$ module, and so C-M of dimension $c-1$.

Assume $n>0$ and the result holds for $n-1$. We prove it for n. There is an exact sequence of complexes

$$0 \to K_{n-1} \to C_{n-1} \to C_n \to 0$$

and so a long exact sequence

$$0 \to H^{d-n}(C_n) \to H^{d-n+1}(K_{n-1}) \xrightarrow{\beta} H^{d-n+1}(C_{n-1})$$

$$\xrightarrow{\alpha} H^{d-n+1}(C_n) \to H^{d-n+2}(K_{n-1}) \to \cdots.$$

$H^{d-n+2}(K_{n-1})=0$ from the corollaries and β is surjective from Lemma 2.5, and so $\alpha=0$. It follows that $H^i(C_n)=0$ if $i \neq d-n$. Since $H^{d-n+1}(K_{n-1})$ is 0 or a C-M module of dimension $n-1+c$ by the corollaries, and since $H^{d-n+1}(C_n)$ is 0 or a C-M module of dimension $n-2+c$ by the inductive hypothesis, it follows that $H^{d-n}(C_n)$ is 0 or a C-M module of dimension $n-1+c$.

Proof of Theorems 1.1 *and* 1.2. From the definition of C_d, there is an exact sequence

$$0 \to R\Big/\bigcap I_j \to \bigoplus R/I_j \to H^0(C_d) \to 0.$$

Using Lemma 2.6 with $n=d$, $H^0(C_d)$ is 0 or C-M of dimension $d+c-1$. Since each R/I_j is C-M of dimension $d+c$, it follows that $R/\bigcap I_j$ is C-M of dimension $d+c$.

REFERENCES

[EH] J. A. EAGON AND M. HOCHSTER, R-sequences and indeterminates, *Quart. J. Math. Oxford Ser.* (2) **25** (1974), 61–71.

[H] M. HOCHSTER, Cohen–Macaulay rings, combinatorics, and simplicial complexes: Ring Theory II, *in* "Proceedings Second Conf., Univ. Oklahoma, Norman, 1975," pp. 171–223, Lecture Notes in Pure and Appl. Math., Vol. 26, Dekker, New York, 1977.

[M] J. R. MUNKRES, Topological results in combinatorics. *Michigan Math. J.* **31** (1984), 113–128.

[R] G. A. REISNER, Cohen–Macaulay quotients of polynomial rings. *Adv. in Math.* **21**, No. 1 (1976), 30–49.

[Sta] R. P. STANLEY, Cohen–Macaulay rings and constructible polytopes. *Bull. Amer. Math. Soc.* (*N.S.*) **81** (1975), 133–135.